Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 155 (2008) 190–200
Topology and its Applications
www.elsevier.com/locate/topol

Remote points in products under CH

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Received 21 October 2006; received in revised form 30 May 2007; accepted 30 May 2007

Abstract

Assume CH. Let I be any index set, and let X_i, for $i \in I$, be a completely regular ccc topological space of weight ω_2. If $X = \prod_{i \in I} X_i$ is ccc and non-pseudocompact, then X has remote points.

MSC: 54D35; 06E10; 03E35

Keywords: Remote points; ccc Boolean algebras

1. Introduction

Let X be a completely regular topological space and let βX denote the Stone–Čech compactification of X. A point p in the remainder $\beta X \setminus X$ is called a *remote point* of X if $p \notin \mathrm{cl}_{\beta X}(D)$ for any nowhere-dense subset D of X. Fine and Gillman [7] proved that under CH, every separable non-pseudocompact space has remote points. Dow [6] showed that it was consistent that there is a separable non-pseudocompact space with no remote points. We are interested in determining whether CH implies that every *ccc* non-pseudocompact space has remote points. It is provable in ZFC that every ccc, non-pseudocompact space of π-weight ω or ω_1 has remote points (the first result was proved independently by Chae and Smith [2] and van Douwen [9], the second by Dow [5]). In [1] it was shown that assuming the existence of an ω_1-scale, if X is a ccc, non-pseudocompact product of ccc spaces each of weight ω_1, then X has a remote point. It was also shown in [1] that under CH, every ccc non-pseudocompact space of weight ω_2 has remote points.

Here we combine some of the techniques used in [1] to show that under CH, if X is a ccc, non-pseudocompact product of ccc spaces each of weight ω_2, then X has a remote point. We will find remote points by constructing remote filters.

Definition 1.1. A collection $\mathcal{F}$ of closed sets in a space X is *remote* if for each nowhere-dense $D \subseteq X$, there is an $F \in \mathcal{F}$ such that $F \cap D = \emptyset$. A *remote filter* is a filter of closed sets that is also remote. If we have written $X = \sum_{m<\omega} X_m$, a remote filter $\mathcal{F}$ is called *nice* if $|\{m < \omega\colon\ F \cap X_m = \emptyset\}| < \omega$ for all $F \in \mathcal{F}$ and if $\bigcap \mathcal{F} = \emptyset$.

See [5] for a proof of the following fact:

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.05.020

Lemma 1.2. *Suppose $\{U_m\colon m<\omega\}$ is a locally finite family of regular closed sets in X, and let B_m be a base for U_m, for each $m<\omega$. Let $\mathcal{A}$ denote the collection of all maximal cellular families of sets from $\bigcup_{m<\omega} B_m$, and let $\mathcal{F}$ be a filter of closed sets in X.*

Suppose that for every $A\in\mathcal{A}$, there are an $F\in\mathcal{F}$ and finite subsets $A_m\subseteq A\cap\mathcal{P}(B_m)$ (for $m<\omega$) such that $F\subseteq\bigcup\{\bar{a}\colon a\in\bigcup_{m<\omega}A_m\}$. Then $\mathcal{F}$ is a remote filter on X and $\mathcal{F}$ extends to a remote point of X.

The technique that we use to find remote filters—a technique first used by van Mill [10] and van Douwen [9], and used repeatedly since then—will be to first define a careful indexing of the basic open sets in a space X. Such an indexing induces a sort of hierarchy on the collection of maximal pairwise-discrete families of basic open sets in X. This hierarchy will determine how to go about choosing the finite sets A_m needed in Lemma 1.2.

1.1. Notation and conventions

Assume CH throughout. All topological spaces are assumed to be completely regular. ccc means "countable chain condition", and a space X is ccc if every pairwise-disjoint collection of open sets in X is countable. If f and g are functions from ω to ω, $f<^* g$ means that $f(m)<g(m)$ for all but finitely-many $m<\omega$. For θ a regular cardinal, $H(\theta)$ is the set of all sets whose transitive closure is of size less than θ. If X and Y are sets, "$X\subseteq Y$" means that X is any subset of Y, whereas "$X\subset Y$" means that X is a proper subset of Y.

If $b=\prod_{i\in I}b(i)$ is a basic open set in the product space $\prod_{i\in I}X_i$, we denote by $\mathrm{supp}(b)$ the support of b—that is, the finite set $\{i\in I\colon b(i)\neq X_i\}$. For A a countable collection of basic open sets in X, $\mathrm{supp}(A)$ will denote the countable set $\bigcup\{\mathrm{supp}(a)\colon a\in A\}$. If b is a basic open set and A is a countable set of basic open sets, the *projection* of b onto $\mathrm{supp}(A)$ is defined by $\mathrm{pr}_A(b)=\prod_{i\in I}\mathrm{pr}_A(b)(i)$, where

$$\mathrm{pr}_A(b)(i)=\begin{cases} b(i) & \text{if } i\in\mathrm{supp}(A),\\ X_i & \text{otherwise.}\end{cases}$$

2. Machinery

Let I be any index set, and for $i\in I$, let X_i be a ccc space of weight ω_2. Let B_i be a base for X_i. Set $X=\prod_{i\in I}X_i$, and assume that X is ccc and non-pseudocompact. Since we are assuming CH, $|RO(X_i)|=\omega_2$; so we may as well assume that $B_i=RO(X_i)$ for each $i\in I$. In what follows, the operations $\bigvee$ and $\bigwedge$ will refer to the Boolean algebra operations on $RO(X_i)$ when they are taking place within an individual space X_i. (For background on Boolean algebras, see [8].) Fix a discrete family $\{U_m\colon m<\omega\}$ of basic open sets in X. (This is possible since X is non-pseudocompact.) Also fix an ω_1-scale $\{f_\xi\colon \xi<\omega_1\}$. (That is, $\{f_\xi\colon\xi<\omega_1\}\subseteq{}^\omega\omega$ such that (1) if $\delta<\xi<\omega_1$ then $f_\delta<^* f_\xi$, and (2) for any $g\in{}^\omega\omega$, there is a $\xi<\omega_1$ such that $g<^* f_\xi$.) We will say "A is a maximal antichain" to abbreviate "A is a maximal pairwise-disjoint collection of nonempty basic open subsets of X, each being a subset of some U_m". Since X is ccc, we may write a maximal antichain A as $\{a(m,l)\colon m,l<\omega\}$, where for $m,l<\omega$, $a(m,l)\subseteq U_m$.

2.1. Envelopes and cinch-sets

In [1], we developed tools for analyzing the structure of a single ccc Boolean algebra of size κ, for κ a regular cardinal. We modify these notions for the present case, in which we are dealing with an arbitrary product of ccc topological spaces each of weight ω_2. For now, assume that for each $i\in I$, the base $B_i=RO(X_i)$ has been written as an increasing union: $B_i=\bigcup_{\alpha<\omega_2}B_i(\alpha)$, where for each $\alpha<\omega_2$, $B_i(\alpha)$ is a σ-complete subalgebra of B_i.

Definition 2.1. Let $i\in I$; let $b_i\in B_i$; and let $\alpha<\omega_2$. The *α-envelope* of b_i is

$$\pi_\alpha(b_i)=\bigwedge\bigl\{c\in B_i(\alpha)\colon b_i\subseteq c\bigr\}.$$

Now, if $\alpha<\omega_2$ and $b=\prod_{i\in I}b(i)$ is a basic open set in X, define the *α-envelope* of b to be

$$\pi_\alpha(b)=\prod_{i\in I}\pi_\alpha\bigl(b(i)\bigr).$$

Note that for all $\alpha < \omega_2$, $\mathrm{supp}(b) \supseteq \mathrm{supp}(\pi_\alpha(b))$.

Let $b = \prod_{i \in I} b(i)$ be a basic open set. For each i in the finite set $\mathrm{supp}(b)$, let $\alpha_i < \omega_2$ be minimal such that $\pi_{\alpha_i}(b(i)) = b(i)$. (Such an α_i exists: since $b(i) \in B_i$, $b(i) \in B_\alpha(i)$ for some α; and whenever $b(i) \in B_i(\alpha)$, $\pi_\alpha(b(i)) = b(i)$.) Define the *level* of b by $l(b) = \max\{\alpha_i\colon i \in \mathrm{supp}(b)\}$. It is readily verified that $l(b) = \min\{\alpha < \omega_2\colon \pi_\alpha(b) = b\} = \min\{\alpha < \omega_2\colon \forall i \in I, \pi_\alpha(b(i)) = b(i)\} = \min\{\alpha < \omega_2\colon \forall \beta \geqslant \alpha, \pi_\beta(b) = b\}$.

The proof of the following lemma is similar to that of Lemma 2.3 in [1].

Lemma 2.2. *Let a, b, c be basic open sets in X and let $\alpha, \beta < \omega_2$. Then*:

(1) $\{\pi_\alpha(a)\colon \alpha < \omega_2\}$ *is a decreasing, eventually-constant sequence of basic open sets in X.*
(2) *If $a \subseteq b$ then $\pi_\alpha(a) \subseteq \pi_\alpha(b)$.*
(3) $\pi_\alpha(\pi_\alpha(a)) = \pi_\alpha(a)$.
(4) *If $\alpha < \beta$ then $\pi_\alpha(b) = \pi_\alpha(\pi_\beta(b))$.*
(5) *If $l(a), l(c) \leqslant \alpha$ and $a \cap b \subseteq c$, then $a \cap \pi_\alpha(b) \subseteq c$.*
(6) *$a \neq \emptyset$ if and only if there is an $\alpha < \omega_2$ such that $\pi_\alpha(a) \neq \emptyset$.*

Definition 2.3. Let b be a basic open set in X. The *cinch-set* of b is

$$\sigma(b) = \{\alpha < \omega_2\colon \pi_\beta(b) \supset \pi_\alpha(b) \text{ for } \beta < \alpha\} = \{\alpha < \omega_2\colon l(\pi_\alpha(b)) = \alpha\}.$$

Note that the cinch-set of b is the set of those α where, in any component, the α-envelope of b suddenly gets "tighter". It will be convenient later to define cinch-sets of basic open sets in a component space X_i as well: if $b(i) \in B_i$, define $\sigma(b(i)) = \{\alpha < \omega_2\colon \pi_\beta(b(i)) \supset \pi_\alpha(b(i)) \text{ for } \beta < \alpha\}$.

Definition 2.4. Let A be a countable collection of basic open sets in X. Define the *cinch-set* of A as

$$\sigma(A) = \overline{\bigcup\{\sigma(a)\colon a \in A\}}.$$

Let $b \subseteq X$ be a basic open set and let $i \in \mathrm{supp}(b)$. By Lemma 2.5 in [1], $\sigma(b(i)) = \{\alpha < \omega_2\colon \forall \beta < \alpha,\ \pi_\beta(b(i)) \supset \pi_\alpha(b(i))\}$ is a countable closed subset of ω_2. If $i \notin \mathrm{supp}(b)$, then $\sigma(b(i)) = \sigma(X_i) = \{0\}$. Thus we have:

Corollary 2.5. *For any countable collection A of basic open sets in X, $\sigma(A)$ is a countable closed subset of ω_2.*

Definition 2.6. Let a, b be basic open sets in X (or in some component space X_i). The *highest common cinching point* (hccp) of a and b is defined to be

$$\mathrm{hccp}(a, b) = \max\big(\sigma(a) \cap \sigma(b)\big).$$

(Note that this is really a "max", and not just a "sup", because $\sigma(a)$ and $\sigma(b)$ are closed in ω_2.)

Also, if A and C are countable collections of basic open sets in X, define

$$\mathrm{hccp}(A, C) = \max\big(\sigma(A) \cap \sigma(C)\big).$$

The following lemma is similar to Corollary 2.13 in [4].

Lemma 2.7. *Let $a_0, \ldots, a_{n-1}$ be basic open sets in X. Then*

$$\sigma\Big(\bigcap_{l<n} a_l\Big) \subseteq \Big(\bigcup_{l<n-1} \mathrm{hccp}\Big(a_l, \bigcap_{l<j<n} a_j\Big)\Big) \cup \Big(\bigcup_{l<n} \sigma(a_l)\Big).$$

In particular, for any basic open sets a and b in X, $\sigma(a \cap b) \subseteq \sigma(a) \cup \sigma(b) \cup \mathrm{hccp}(a, b)$.

Definition 2.8. Let $a, b \subseteq X$ be basic open sets in X (or in some X_i). Set $\alpha = \mathrm{hccp}(a, b)$. We say b is *weakly below* a, and write $b \subseteq_w a$ or $a \supseteq_w b$, if $\pi_\alpha(b) \subseteq \pi_\alpha(a)$.

Note. By parts (1), (3), and (4) of Lemma 2.2, $b \subseteq_w a$ if and only if $b \subseteq \pi_\alpha(a)$. By part (3) of Lemma 2.2, if $b \subseteq a$ then $b \subseteq_w a$.

Definition 2.9. A (finite) sequence $\{a_p\colon\ p < k\}$ of basic open sets in X (or in some X_i) is called *weakly descending* if for all $p < q < k$, $a_p \supseteq_w a_q$.

The proof of the following lemma is similar to that of Lemma 2.17 in [1].

Lemma 2.10. *Let $\alpha < \omega_2$ and let $\{a_p\colon\ p < k\}$ be a weakly-descending sequence of nonempty basic open sets in X (or in some X_i). Then*

(i) $\bigcap_{p<k} \pi_\alpha(a_p) = \pi_\alpha(\bigcap_{p<k} a_p)$;
(ii) $\sigma(\bigcap_{p<k} a_p) \subseteq \bigcup_{p<k} \sigma(a_p)$; *and*
(iii) $\bigcap_{p<k} a_p > 0$.

2.2. Organization of the bases

In the following definition, we organize each base B_i with the help of elementary submodels as in [1]. It is possibly intuitively helpful (although not completely accurate) to imagine B_i as being divided-up into an ω_1-by-ω_2 matrix. The "columns" $B_i(\alpha)$, for $\alpha < \omega_2$, will be ω_1-sized subalgebras of B_i. The entries in each column $B_i(\alpha)$ will be countable subcollections $B_i(\alpha, \xi)$ of $B_i(\alpha)$, for $\alpha < \omega_1$. The αth column will contain all previous columns and their entries. Each countable entry $B_i(\alpha, \xi)$ in the αth column will also, in some sense, "know about" previous columns and their entries.

See [3] for an introduction to elementary submodels and their use in topology.

Definition 2.11. For $i \in I$, inductively define $\{M_i(\alpha)\colon\ \alpha < \omega_2\}$ and $\{\{M_i(\alpha, \xi)\colon\ \xi < \omega_1\}\colon\ \alpha < \omega_2\}$ such that for every $\alpha < \omega_2$,

(1) $M_i(\alpha)$ is an elementary submodel of $H(\theta)$ (for some fixed sufficiently-large regular cardinal θ) and $\{\alpha, B_i, \{M_i(\beta)\colon\ \beta < \alpha\}, \{\{M_i(\beta, \xi)\colon\ \xi < \omega_1\}\colon\ \beta < \alpha\}\} \in M_i(\alpha)$;
(2) For $\lambda < \omega_2$ with $\mathrm{cf}(\lambda) = \omega_1$, $M_i(\lambda) = \bigcup_{\alpha<\lambda} M_i(\alpha)$;
(3) $|M_i(\alpha)| = \omega_1$ and $[M_i(\alpha)]^\omega \subseteq M_i(\alpha)$;
(4) $\{M_i(\alpha, \xi)\colon\ \xi < \omega_1\}$ is a continuous elementary ϵ-chain of countable elementary submodels of $M_i(\alpha)$;
(5) $\bigcup\{M_i(\alpha, \xi)\colon\ \xi < \omega_1\} = M_i(\alpha)$; and
(6) $I, \{X_i\colon\ i \in I\}, \{B_i\colon\ i \in I\}, i, X_i, B_i, \{U_m\colon\ m < \omega\}, \{M_i(\beta)\colon\ \beta < \alpha\}$, and $\{M_i(\beta, \xi)\colon\ \beta < \alpha, \xi < \omega_1\}$ are elements of $M_i(\alpha, 0)$.

Set $B_i(0) = M_i(0) \cap B_i$. For $\alpha < \omega_2$ a successor ordinal, set $B_i(\alpha) = M_i(\alpha) \cap B_i$. For $\lambda < \omega_2$ a limit ordinal, set $B_i(\lambda) = \bigcup\{B_i(\alpha)\colon\ \alpha < \lambda\}$.

Note that since each $B_i = RO(X_i)$ is complete and ccc, condition (3) above implies that for each $i \in I$ and $\alpha < \omega_2$, $M_i(\alpha) \cap RO(X_i)$ is a complete subalgebra of $RO(X_i)$ of size ω_1. Note also that we have really invoked CH here to know that such complete ω_1-sized subalgebras can be constructed.

If M is any elementary submodel of $H(\theta)$ (for some regular θ), and if $C \in M$ is a countable set, then $C \subseteq M$ (see [3] for a proof of this). From this, and by Lemma 2.5, it follows that:

Lemma 2.12. *Suppose C is a countable collection of open sets in X and M is any elementary submodel of $H(\theta)$ (for some regular θ) such that $I, \{X_i\colon\ i \in I\}, \{B_i\colon\ i \in I\}, \{M_i(\alpha)\colon\ i \in I, \alpha < \omega_2\}, \{M_i(\alpha, \xi)\colon\ \alpha < \omega_2, \xi < \omega_1\}$, and C are elements of M. Then $\sigma(C) \in M$, and also $\sigma(C) \subseteq M$.*

2.3. Norms and denseness

We define a notion of "norm" for each basic open set $b \subseteq X$; this will be the indexing that will help to induce the required hierarchy on the collection of maximal antichains in X.

Definition 2.13. For each $\omega \leqslant \delta < \omega_1$, fix a bijection $g_\delta : \omega \to \delta$; and for each $n < \omega$, let $g_n : \omega \to n$ be any surjection that is eventually 0.

For $\delta < \omega_1$ and $n < \omega$, set

$$E_{\delta,n} = \{g_\delta(k):\ k < n\}$$

(that is, the "first" n-many ordinals in δ according to g_δ, if δ is infinite).

For each $i \in I$, $\alpha < \omega_2$, and $\xi < \omega_1$, fix a listing of the countable set $M_i(\alpha,\xi) \cap B_i$ in order type ξ: $M_i(\alpha,\xi) \cap B_i = \{b_{i\alpha\xi}(\delta):\ \delta < \xi\}$.

Definition 2.14. Let $b = \prod_{i\in I} b(i)$ be a basic open set in X. Let $\alpha < \omega_2$, $\xi < \omega_1$, and $n < \omega$. Say that (α,ξ)-norm$(b) \leqslant n$ if the following hold:

(1) $|\operatorname{supp}(b)| \leqslant n$;
(2) For all $i \in \operatorname{supp}(b)$, $b(i) \in M_i(\alpha,\xi)$; and (moreover)
(3) For all $i \in \operatorname{supp}(b)$, $b(i) \in \{b_{i\alpha\xi}(\delta):\ \delta \in E_{\xi,n}\}$.

Before proving the next lemma, we observe that the following hold by the construction in Definition 2.11: (i) for each $i \in I$, $B_i = \bigcup_{\alpha<\omega_2} B_i(\alpha)$; (ii) if $\alpha < \beta < \omega_2$ and $i \in I$ then $M_i(\alpha) \subseteq M_i(\beta)$; and (iii) if $\alpha < \omega_2$, $\xi < \zeta < \omega_1$, and $i \in I$, then $M_i(\alpha,\xi) \subseteq M_i(\alpha,\zeta)$.

Note that if b is any basic open set, there are $\alpha < \omega_2$, $\xi < \omega_1$, and $n < \omega$ such that (α,ξ)-norm$(b) \leqslant n$, because $\operatorname{supp}(b)$ is a finite subset of I.

Lemma 2.15. *Let a be a basic open set in X, and let $A = \{a_m:\ m < \omega\}$ be a countable collection of basic open sets in X.*

(1) *Let $\alpha < \omega_2$, $\xi < \omega_1$, and $n < \omega$. If (α,ξ)-norm$(a) \leqslant n$ then (α,ξ)-norm$(\mathrm{pr}_A(a)) \leqslant n$.*
(2) *Let $\alpha < \omega_2$, $\xi < \omega_1$, and $n < n' < \omega$. If (α,ξ)-norm$(a) \leqslant n$, then (α,ξ)-norm$(a) \leqslant n'$.*
(3) *There exist $\alpha < \omega_2$ and $\xi < \omega_1$ such that for all $m < \omega$ and all $i \in I$, $a_m(i) \in M_i(\alpha,\xi)$.*
(4) *Let $\alpha < \omega_2$ and $\xi < \omega_1$. If $a_m(i) \in M_i(\alpha,\xi)$ for all $m < \omega$ and $i \in I$, then there exists a $p \in {}^{\omega}\omega$ such that for each $m < \omega$, (α,ξ)-norm$(a_m) \leqslant p(m)$.*

Proof. Part (1) is clear from Definition 2.13 and the definition of projections.

Noting that $n < n' \Rightarrow E_{\xi,n} \subseteq E_{\xi,n'}$, part (2) also follows from Definition 2.13.

For (3): Let $m < \omega$ and $i \in \operatorname{supp}(a_m)$. Since $a_m(i) \in B_i = \bigcup_{\alpha<\omega_2} B_i(\alpha)$, there is an $\alpha_{mi} < \omega_2$ such that $a_m(i) \in B_i(\alpha_{mi})$. Let $\alpha_A = \sup\{\alpha_{mi}:\ m < \omega, i \in \operatorname{supp}(a_m)\}$. $\alpha_A < \omega_2$, as A is countable and each $\operatorname{supp}(a_m)$ is finite. Also, if (for some $m < \omega$ and $i \in I$) $i \notin \operatorname{supp}(a_m)$, then $a_m(i) = X_i \in B_i(\alpha)$ for any $\alpha < \omega_2$. Thus for every $m < \omega$ and $i \in I$, $a_m(i) \in B_i(\alpha_{mi}) \subseteq B_i(\alpha_A) \subseteq M_i(\alpha_A)$. Next: as $M_i(\alpha_A) = \bigcup_{\xi<\omega_1} M_i(\alpha_A,\xi)$, for each $m < \omega$ and $i \in \operatorname{supp}(a_m)$ there is a $\xi_{mi} < \omega_1$ such that $a_m(i) \in M_i(\alpha_A,\xi_{mi})$. Set $\xi_A = \sup\{\xi_{mi}:\ m < \omega, i \in I\}$. $\xi_A < \omega_1$ as $\{\xi_{mi}:\ m < \omega, i \in I\}$ is countable. If $i \notin \operatorname{supp}(a_m)$ for some m, then $a_m(i) = X_i \in M_i(\alpha_A,\xi)$ for any $\xi < \omega_1$. Thus for every $m < \omega$ and $i \in I$, $a_m(i) \in M_i(\alpha_A,\xi_{mi}) \subseteq M_i(\alpha_A,\xi_A)$.

For (4): Let $\alpha < \omega_2$ and $\xi < \omega_1$, and suppose $a_m(i) \in M_i(\alpha,\xi)$ for all $m < \omega$ and $i \in I$. Let $m < \omega$ and $i \in I$. Because $a_m(i) \in M_i(\alpha,\xi) \cap B_i = \{b_{i\alpha\xi}(\delta):\ \delta < \xi\} = \{b_{i\alpha\xi}(\delta):\ \delta \in \bigcup_{n<\omega} E_{\xi,n}\}$, and because $\operatorname{supp}(a_m)$ is finite, there is a $p_1(m) < \omega$ such that $a_m(i) \in \{b_{i\alpha\xi}(\delta):\ \delta \in E_{\xi,p_1(m)}\}$ for each $i \in \operatorname{supp}(a_m)$. Also since $|\operatorname{supp}(a_m)| < \omega$, choose a $p_2(m) < \omega$ such that $|\operatorname{supp}(a_m)| \leqslant p_2(m)$. Define $p \in^{\omega} \omega$ by $p(m) = \max\{p_1(m), p_2(m)\}$. Then for each $m < \omega$, (α,ξ)-norm$(a_m) \leqslant p(m)$. $\square$

Recall, for the next definition, that we have fixed a discrete family $\{U_m:\ m < \omega\}$ of basic open sets in X.

Definition 2.16. Let A be a family of basic open sets in X. Let $\alpha < \omega_2$, $\xi < \omega_1$, and $n, k, m < \omega$. Say that A is *(α, ξ, n, k)-dense in U_m* if for any k-many basic open sets $\{b_l\colon l < k\}$ each of (α, ξ)-norm $\leqslant n$, if $(\bigcap_{l<k} b_l) \cap U_m \neq \emptyset$, then $(\bigcup A) \cap (\bigcap_{l<k} b_l) \cap U_m \neq \emptyset$.

Since X is completely regular, any maximal antichain A will always be (α, ξ, n, k)-dense in U_m for any α, ξ, n, k, and m.

Lemma 2.17. *Let A be a family of basic open sets in X, and suppose that A is (α, ξ, n, k)-dense in U_m for some $\alpha < \omega_2$, $\xi < \omega_1$, and $n, k, m < \omega$. Then there is a finite subset A' of A that is also (α, ξ, n, k)-dense in U_m.*

Proof. Let A be as described. Without loss of generality, $a \subseteq U_m$ for each $a \in A$.

Let A_0 be any nonempty finite subset of A.

Suppose finite subsets $A_0, A_1, \ldots, A_j$ of A have been chosen, for some $j \leqslant nk$. Denote $I_j = \operatorname{supp}(A_j)$. Let D_j denote the k-tuples $\vec{b} = \{b_l\colon l < k\}$ of basic open sets such that for each $l < k$, $\operatorname{supp}(b_l) \subseteq I_j$ and (α, ξ)-norm$(b_l) \leqslant n$. D_j is a finite set; this follows from the fact that I_j and $E_{\xi,n}$ are finite sets. Let $\vec{b} \in D_j$. Since A is (α, ξ, n, k)-dense in U_m, if $(\bigcap \vec{b}) \cap U_m \neq \emptyset$ then there is an $a_{\vec{b}} \in A$ such that $(\bigcap \vec{b}) \cap a_{\vec{b}} \neq \emptyset$. Define $A_{j+1} = A_j \cup \{a_{\vec{b}}\colon \vec{b} \in D_j$ and $(\bigcap \vec{b}) \cap U_m \neq \emptyset\}$.

In this way, generate an increasing sequence of finite subsets $A_0, A_1, \ldots, A_{nk+1}$ of A; we will show that the last of these, A_{nk+1}, is (α, ξ, n, k)-dense in U_m.

Claim. *For any k-tuple $\vec{b} = \{b_l\colon l < k\}$ of basic open sets in X with (α, ξ)-norm$(b_l) \leqslant n$ for each $l < k$, if $(\bigcap \vec{b}) \cap U_m \neq \emptyset$ then there is a $j \leqslant nk$ so that $\operatorname{supp}(\bigcap \vec{b}) \cap I_j = \operatorname{supp}(\bigcap \vec{b}) \cap I_{j+1}$.*

Proof. Suppose otherwise. Then $\emptyset \neq \operatorname{supp}(\bigcap \vec{b}) \cap I_1 \subset \operatorname{supp}(\bigcap \vec{b}) \cap I_2 \subset \cdots \subset \operatorname{supp}(\bigcap \vec{b}) \cap I_{nk} \subset \operatorname{supp}(\bigcap \vec{b}) \cap I_{nk+1}$. Then since $|\operatorname{supp}(\bigcap \vec{b}) \cap I_1| \geqslant 1$, it must be that $|\operatorname{supp}(\bigcap \vec{b}) \cap I_{nk+1}| \geqslant nk + 1$; but this is a contradiction: we have assumed that for all $l < k$, (α, ξ)-norm$(b_l) \leqslant n$; so, in particular, $|\operatorname{supp}(b_l)| \leqslant n$; and thus $|\operatorname{supp}(\bigcap \vec{b}) \cap I_{nk+1}|$ should be at most nk. This proves the Claim. □

Now: let $\{b_l\colon l < k\}$ be any k-tuple of basic open sets in X such that (α, ξ)-norm$(b_l) \leqslant n$ for $l < k$ and $(\bigcap_{l<k} b_l) \cap U_m \neq \emptyset$. By the Claim, choose $j \leqslant nk$ such that $\operatorname{supp}(\bigcap_{l<k} b_l) \cap I_j = \operatorname{supp}(\bigcap_{l<k} b_l) \cap I_{j+1}$. Set $b'_l = \operatorname{pr}_{A_j}(b_l)$ for each $l < k$. As $b_l \subseteq b'_l$, $(\bigcap_{l<k} b'_l) \cap U_m \neq \emptyset$. Also, since $\operatorname{supp}(\bigcap_{l<k} b'_l) \subseteq I_j$, by Lemma 2.15 we have $\{b'_l\colon l < k\} \in D_j$. By construction of A_{j+1}, there is an $a \in A_{j+1}$ such that $a \cap (\bigcap_{l<k} b'_l) \neq \emptyset$. We claim that $a \cap (\bigcap_{l<k} b_l) \neq \emptyset$: for suppose not. Then there is an $i \in I$ such that $a(i) \cap (\bigcap_{l<k} b_l)(i) = \emptyset$. Then $i \in \operatorname{supp}(a) \subseteq I_{j+1}$ and $i \in \operatorname{supp}(\bigcap_{l<k} b_l)$; so by choice of j, $i \in \operatorname{supp}(\bigcap_{l<k} b_l) \cap I_j \subseteq I_j$. But then

$$\emptyset = a(i) \cap \left(\bigcap_{l<k} b_l\right)(i) = a(i) \cap \left(\bigcap_{l<k} b_l(i)\right) = a(i) \cap \left(\bigcap_{l<k} b'_l(i)\right)$$

so that $a \cap (\bigcap_{l<k} b'_l) = \emptyset$, contradicting the choice of a.

Thus for any k-tuple $\vec{b}$ with (α, ξ)-norm$(b_l) \leqslant n$ for $l < k$ and $(\bigcap \vec{b}) \cap U_m \neq \emptyset$, there is an $a \in A_{nk+1}$ such that $a \cap (\cap \vec{b}) \cap U_m \neq \emptyset$ (since the A_j's are increasing). Thus A_{nk+1} is a finite subset of A that is also (α, ξ, n, k)-dense with respect to U_m. □

2.4. A few more elementary submodels

Let A be a maximal antichain in X. Let $M_0^A \in M_1^A \in \cdots \in M_n^A \in \cdots$ be an elementary $\in$-chain of countable elementary submodels of $H(\theta)$, for some fixed sufficiently-large θ, where we have put $A, X, I, \{f_\xi\colon \xi < \omega_1\}, \{g_\delta\colon \delta < \omega_1\}$, $\{\{M_i(\alpha)\colon \alpha < \omega_2\}\colon i \in I\}$, and $\{\{\{M_i(\alpha, \xi)\colon \xi < \omega_1\}\colon \alpha < \omega_2\}\colon i \in I\}$ into M_0^A. Set $\xi_n^A = M_n^A \cap \omega_1$, for $n < \omega$. Set $M^A = \bigcup_{n<\omega} M_n^A$. Let $\xi^A = M^A \cap \omega_1$.

The ordinal ξ^A is determined in part by the (α, ξ)-norms of open sets a contained in the maximal antichain A. In the next section, we will rank maximal antichains A according to their associated ξ^A.

By elementarity and since $\{f_\xi\colon \xi < \omega_1\}$ is a scale, we have:

Lemma 2.18. *Let A be a maximal antichain and let $n < \omega$. Then for all $p \in {}^\omega\omega \cap M_n^A$, $p <^* f_{\xi_n^A}$. Also if $p \in {}^\omega\omega \cap M^A$ then $p <^* f_{\xi^A}$.*

3. Main result

Definition 3.1. For each maximal antichain $A = \{a(m,l)\colon\ m, l < \omega\}$ in X, define

$$F_A = \bigcup_{m<\omega}\left(\bigcup_{l<f_{\xi^A}(m)} a(m,l)\right).$$

Then define $\mathcal{F} = \{F_A\colon\ A \text{ is a maximal antichain}\}$.

We will show that $\mathcal{F}$ generates a nice remote filter on X with respect to $\{U_m\colon\ m < \omega\}$. The niceness and remoteness will hold by construction and by Lemma 1.2. We need to verify that for all $n < \omega$ and all maximal antichains $C^0, \dots, C^{n-1}$ in X, $[\bigcap\{F^{C^k}\colon\ k < n\}] \cap U_m \neq \emptyset$ for almost all m. The following lemma will allow us to produce sequences $\{b_m^k\colon\ m < \omega\}$ of open sets in X, for $k < n$, such that $\emptyset \neq \bigcap_{k<n} b_m^k \subseteq [\bigcap\{F^{C^k}\colon\ k < n\}] \cap U_m$ for almost all m.

Lemma 3.2. *Let $k < \omega$ and let $C^0, C^1, \dots, C^k$ be maximal antichains labeled so that $\xi^{C^0} \leqslant \xi^{C^1} \leqslant \dots \leqslant \xi^{C^k}$. Suppose we are given sequences $\{b_m^p\colon\ m < \omega\}$ of basic open sets in X, for each $p < k$, such that the following hold:*

1. *For all $p < k$, there exists $N_p < \omega$ such that $\{b_m^p\colon\ m < \omega\} \in M_{N_p}^{C^p}$;*
2. *For all $p < k$, there exists $S_p < \omega$ such that for all $m \geqslant S_p$, there exists $l < f_{\xi_{N_p+2}^{C^p}}(m)$ such that $\emptyset \neq b_m^p \subseteq c_p(m,l)$;*
3. *For each increasing subsequence $\{p_r\colon\ r < s\}$ of $\{0, \dots, k-1\}$, each $m < \omega$, and each $i \in I$ such that $b_m^{p_r}(i) \neq X_i$ for all $r < s$, $\{b_m^{p_r}(i)\colon\ r < s\}$ is a weakly-descending sequence; and*
4. *For all $m < \omega$, $\bigcap_{p<k} b_m^p \neq \emptyset$.*

Then we can find a sequence $\{b_m^k\colon\ m < \omega\}$ of basic open sets in X such that

1. *There exists $N_k < \omega$ with $N_k \geqslant \max\{N_p\colon\ p < k\}$ such that $\{b_m^k\colon\ m < \omega\} \in M_{N_k}^{C^k}$;*
2. *There exists $S_k < \omega$ with $S_k \geqslant \max\{S_p\colon\ p < k\}$ such that for all $m \geqslant S_k$, there exists $l < f_{\xi_{N_k+2}^{C^k}}(m)$ such that $\emptyset \neq b_m^k \subseteq c_k(m,l)$;*
3. *For each increasing subsequence $\{p_r\colon\ r < s\}$ of $\{0, \dots, k\}$, each $m < \omega$, and each $i \in I$ such that $b_m^{p_r}(i) \neq X_i$ for all $r < s$, $\{b_m^{p_r}(i)\colon\ r < s\}$ is a weakly-descending sequence; and*
4. *For all $m < \omega$, $\bigcap_{p\leqslant k} b_m^p \neq \emptyset$.*

Proof. Suppose we are given maximal antichains $\{C^p\colon\ p \leqslant k\}$ and sequences $\{b_m^p\colon\ m < \omega\}$ (for $p < k$) as in the hypotheses of the lemma. Set $A_0 = C^k$. Pick $p_0 < k$ such that $\gamma_0 := \mathrm{hccp}(\{b_m^{p_0}\colon\ m < \omega\}, A_0)$ is maximal. Suppose for some $j < k$ that A_j, p_j, and γ_j have been defined. Set $A_{j+1} = \{\mathrm{pr}_{C_k}(\pi_{\gamma_j}(b_m^{p_j})) \cap a\colon\ m < \omega, a \in A_j\}$. Pick $p_{j+1} \in k \setminus \{p_0, p_1, \dots, p_j\}$ such that $\gamma_{j+1} := \mathrm{hccp}(\{b_m^{p_{j+1}}\colon\ m < \omega\}, \bigcup_{n\leqslant j+1} A_n)$ is maximal.

Claim 1. *For all $j < k$, $\gamma_{j+1} \leqslant \gamma_j$.*

Proof. Suppose by way of contradiction that $\gamma_{j+1} > \gamma_j$ for some $j < k$. Recall that $\gamma_{j+1} = \max(\sigma(\{b_m^{p_{j+1}}\colon\ m < \omega\}) \cap \sigma(\bigcup_{n\leqslant j+1} A_n))$.

Case 1: $\gamma_{j+1} \in \sigma(\bigcup_{n\leqslant j} A_n)$. Then $\gamma_{j+1} \leqslant \gamma_j$ by maximality of γ_j, which is a contradiction.

Case 2: $\gamma_{j+1} \in \sigma(A_{j+1}) = \overline{\bigcup\{\sigma(a)\colon\ a \in A_{j+1}\}}$. Then for every open set O (in the order topology on ω_2) with $\gamma_{j+1} \in O$ and $O \subseteq (\gamma_j, \omega_2)$, there is an $a \in A_{j+1}$ and an $\alpha \in \sigma(a)$ such that $\alpha \in O$. As $a \in A_{j+1}$, there is an $m < \omega$

and an $a' \in A_j$ such that $a = \mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j})) \cap a'$. Then by Lemma 2.7, $\alpha \in \mathrm{hccp}(\mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j})), a') \cup \sigma(\pi_{\gamma_j}(b_m^{p_j}))) \cup \sigma(a')$. It must be that $\alpha \in \sigma(a')$, as we have assumed that $\alpha > \gamma_j$.

Thus every open set O about γ_{j+1} contains a point $\alpha \in \sigma(A_j) \subseteq \sigma(\bigcup_{n \leqslant j} A_n)$, so $\gamma_{j+1} \in \sigma(\bigcup_{n \leqslant j} A_n)$ (by Lemma 2.5). But then $\gamma_{j+1} \in \sigma(\{b_m^{p_{j+1}}\colon m < \omega\}) \cap \sigma(\bigcup_{n \leqslant j} A_n)$, so that $\gamma_{j+1} \leqslant \gamma_j$ by the maximality of γ_j; and this is a contradiction.

This proves Claim 1. □

Claim 2. *For all $j < k$,* $\mathrm{hccp}(\{b_m^{p_j}\colon m < \omega\}, A_k) \leqslant \gamma_j$.

Proof. Let $j < k$, $m < \omega$, and $a \in A_k$. Let $\alpha = \mathrm{hccp}(b_m^{p_j}, a)$. (We prove that $\alpha \leqslant \gamma_j$, and then Claim 2 will follow by definition of the hccp of two sets.) Unraveling a a bit, we may write

$$a = \mathrm{pr}_{C^k}\big(\pi_{\gamma_{k-1}}(b_{m_{k-1}}^{p_{k-1}})\big) \cap \mathrm{pr}_{C^k}\big(\pi_{\gamma_{k-2}}(b_{m_{k-2}}^{p_{k-2}})\big) \cap \cdots \cap \mathrm{pr}_{C^k}\big(\pi_{\gamma_j}(b_{m_j}^{p_j})\big) \cap a'$$

for some $a' \in A_j$. By Claim 1, $\gamma_{k-1}, \gamma_{k-2}, \ldots, \gamma_{j+1} \leqslant \gamma_j$. Thus for $r = k-1, k-2, \ldots, j+1, j$, we have $\sigma(\mathrm{pr}_{C^k}(\pi_{\gamma_r}(b_{m_r}^{p_r}))) \subseteq [0, \gamma_r] \subseteq [0, \gamma_j]$. Then by Lemma 2.7, $\sigma(a) \subseteq [0, \gamma_j] \cup \sigma(a')$. Thus we have $\alpha \in \sigma(b_m^{p_j}) \cap ([0, \gamma_j] \cup \sigma(a'))$. If $\alpha \in [0, \gamma_j]$, we are done; and if $\alpha \in \sigma(b_m^{p_j}) \cap \sigma(a')$, then $\alpha \leqslant \gamma_j$ by definition of γ_j.

This proves Claim 2. □

Claim 3. *There exists $N < \omega$ such that the following are elements of $M_N^{C^k}$: $\{\gamma_j\colon j < k\}$, $\{\{\mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p^j}))\colon m < \omega\}\colon j < k\}$, and $\{A_j\colon j \leqslant k\}$.*

Proof. Recall that $\gamma_0 = \mathrm{hccp}(\{b_m^{p_0}\colon m < \omega\}, A_0) \in \sigma(A_0)$. Since $A_0 = C^k \in M_0^{C^k}$ and (by Lemmas 2.5 and 2.12) $\sigma(A_0)$ is a countable element of $M_0^{C^k}$, $\gamma_0 \in M_0^{C^k}$ since $M_0^{C^k} \prec H(\theta)$. Similarly, $\gamma_0 \in M_{N_{p_0}}^{C^{p_0}}$ since (by assumption) $\{b_m^{p_0}\colon m < \omega\} \in M_{N_{p_0}}^{C^{p_0}}$. By definition of envelopes and by Lemma 2.15,

$$H(\theta) \models \text{“}(\exists \xi < \omega_1)(\forall m < \omega)(\forall i \in I)\big(\pi_{\gamma_0}(b_m^{p_0})(i) \in M_i(\gamma_0, \xi)\big)\text{”}.$$

Since all free variables in this statement are elements of $M_{N_{p_0}}^{C^{p_0}}$, by elementarity there exists such a ξ with $\xi < \xi_{N_{p_0}}^{C^{p_0}}$. Pick $N(0)$ so large that $\xi_{N(0)}^{C^k} \geqslant \xi_{N_{p_0}}^{C^{p_0}}$. (This is possible as we have assumed that $\xi^{C^{p_0}} \leqslant \xi^{C^k}$.) Then since, by Definition 2.11, $M_i(\gamma_0, \xi_{N_{p_0}}^{C^{p_0}}) \subseteq M_i(\gamma_0, \xi_{N(0)}^{C^k})$ for all $i \in I$, we have that

$$(\forall m < \omega)(\forall i \in I)\big(\pi_{\gamma_0}(b_m^{p_0})\big)(i) \in M_i\big(\gamma_0, \xi_{N(0)}^{C^k}\big)\big).$$

Because $M_i(\gamma_0, \xi_{N(0)}^{C^k}) \in M_{N(0)+1}^{C^k}$ whenever $i \in M_{N(0)+1}^{C^k}$, it follows that $\{\mathrm{pr}_{C^k}(\pi_{\gamma_0}(b_m^{p_0}))\colon m < \omega\} \in M_{N(0)+1}^{C^k}$.

Now let $j < k$ and suppose we have found $N(j) < \omega$ such that the following things are elements of $M_{N(j)+1}^{C^k}$: $\{\gamma_n\colon n \leqslant j\}$, $\{\{\mathrm{pr}_{C^k}(\pi_{\gamma_n}(b_m^{p_n}))\colon m < \omega\}\colon n \leqslant j\}$, and $\{A_n\colon n \leqslant j\}$. Then $A_{j+1} = \{\mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j})) \cap a\colon a \in A_j\} \in M_{N(j)+1}^{C^k}$, and $\gamma_{j+1} = \mathrm{hccp}(\{b_m^{p_{j+1}}\colon m < \omega\}, \bigcup_{n \leqslant j+1} A_n) \in \sigma(\bigcup_{n \leqslant j+1} A_n) \in M_{N(j)+1}^{C^k}$. Also $\gamma_{j+1} \in \sigma(\{b_m^{p_{j+1}}\colon m < \omega\})$, so $\gamma_{j+1} \in M_{N_{p_{j+1}}}^{C^{p_{j+1}}}$. Then similarly to the base case, we can find $N(j+1) \geqslant N(j)$ such that $\{\mathrm{pr}_{C^k}(\pi_{\gamma_{j+1}}(b_m^{p_{j+1}}))\colon m < \omega\} \in M_{N(j+1)+1}^{C^k}$. Thus $\{\gamma_n\colon n \leqslant j+1\}$, $\{\{\mathrm{pr}_{C^k}(\pi_{\gamma_n}(b_m^{p_n}))\colon m < \omega\}\colon n \leqslant j+1\}$, and $\{A_n\colon n \leqslant j+1\}$ are elements of $M_{N(j+1)+1}^{C^k}$. Claim 3 follows. □

Claim 4. *There exists $S_k < \omega$ with $S_k \geqslant S_{k-1}$ such that for all $m \geqslant S_k$,*

(1) *For all $j < k$, $(\gamma_0, \xi_N^{C^k})$-norm$(\mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j}))) \leqslant f_{\xi_{N+1}^{C^k}}(m)$, and*

(2) *$\{c_k(m, l)\colon l < f_{\xi_{N+2}^{C^k}}(m)\}$ is $(\gamma_0, \xi_N^{C^k}, f_{\xi_{N+1}^{C^k}}(m), k-1)$-dense in U_m.*

Proof. By Claim 3, for all $j < k$, $\{\mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p^j}))\colon m < \omega\} \in M_N^{C^k}$. Since (by Claim 1) $\gamma_j \leqslant \gamma_0$ for all $j < k$, and since $\gamma_j \leqslant \gamma_0 \Rightarrow M_i(\gamma_j) \subseteq M_i(\gamma_0)$ for all $i \in I$, it follows from Lemma 2.15 that

$$H(\theta) \models \text{“}(\forall j < k)(\forall m < \omega)(\forall i \in I)\big(\mathrm{pr}_{C^k}\big(\pi_{\gamma_j}(b_m^{p_j})\big)(i) \in M_i\big(\gamma_0, \xi_N^{C^k}\big)\text{”}.$$

Then again by Lemma 2.15,

$$H(\theta) \models \text{“}(\exists q \in {}^\omega\omega)(\forall j < k)(\forall m < \omega)\big((\gamma_0, \xi_N^{C^k})\text{-norm}\big(\mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j}))\big) \leqslant q(m)\big)\text{”}.$$

Since, by Claim 3, all parameters here are elements of $M_{N+1}^{C^k}$, there exists such a $q \in {}^\omega\omega \cap M_{N+1}^{C^k}$. Then by Lemma 2.18,

$$(\exists S_k < \omega)(\forall m \geqslant S_k)(\forall j < k)\big((\gamma_0, \xi_N^{C^k})\text{-norm}\big(\mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j}))\big) \leqslant f_{\xi_{N+1}^{C^k}}(m)\big),$$

and we may take $S_k \geqslant S_{k-1}$.

Now for part (2) of Claim 4: since C^k is a maximal antichain, in particular $\{c_k(m,l)\colon l < \omega\}$ is $(\gamma_0, \xi_N^{C^k}, f_{\xi_{N+1}^{C^k}}(m), k-1)$-dense in U_m for every $m < \omega$. By Lemma 2.17,

$$H(\theta) \models \text{“}(\exists q \in {}^\omega\omega)(\forall m < \omega)\big(\big\{c_k(m,l)\colon l < q(m)\big\} \text{ is } (\gamma_0, \xi_N^{C^k}, f_{\xi_{N+1}^{C^k}}(m), k-1)\text{-dense in } U_m\big)\text{”}.$$

Since all free variables in this statement are elements of $M_{N+2}^{C^k}$, there exists such a $q \in {}^\omega\omega \cap M_{N+2}^{C^k}$. Then by Lemma 2.18, part (2) holds.

Thus Claim 4 holds. □

Let $m \geqslant S_k$. By assumption, $(\bigcap_{p<k} b_m^p) \cap U_m = (\bigcap_{j<k} b_m^{p_j}) \cap U_m \neq \emptyset$, so also $(\bigcap_{j<k} \mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j}))) \cap U_m \neq \emptyset$. By Claim 4, there is an $l < f_{\xi_{N+2}^{C^k}}(m)$ such that $c_k(m,l) \cap (\bigcap_{j<k} \mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j}))) \cap U_m \neq \emptyset$. Set $b_m^k = c_k(m,l) \cap (\bigcap_{j<k} \mathrm{pr}_{C^k}(\pi_{\gamma_j}(b_m^{p_j})))$. For $m < S_k$, just define $b_m^k = \prod_{i\in I} X_i$. Note that $b_m^k \in A_k$. By construction, $\{b_m^k\colon m < \omega\}$ satisfies conclusion (2) of the lemma. Since (by Claim 3) such $\{b_m^k\colon m < \omega\}$ was constructed using parameters in $M_{N+3}^{C^k}$, by elementarity such $\{b_m^k\colon m < \omega\}$ can be found in $M_{N+3}^{C^k}$. Thus condition (1) in the conclusion of the lemma is satisfied.

Now we show that, however it is chosen, such a sequence $\{b_m^k\colon m < \omega\}$ must satisfy conclusions (3) and (4) of the lemma.

Claim 5. *For each increasing subsequence $\{p_r\colon r < s\}$ of $\{0, \dots, k\}$, each $m < \omega$, and each $i \in I$ such that $b_m^{p_r}(i) \neq X_i$ for all $r < s$, $\{b_m^{p_r}(i)\colon r < s\}$ is a weakly-descending sequence.*

Proof. Suppose there are an increasing subsequence $\{p_r\colon r < s\}$ of $\{0, \dots, k\}$, an $m < \omega$, and an $i \in I$ such that $b_m^{p_r}(i) \neq X_i$ for all $r < s$. By hypothesis (3) of the lemma, we need only consider the case when $p_{s-1} = k$; and in this case we need only check that for all $r < s-1$, $b_m^{p_r}(i) \supseteq_w b_m^k(i)$. Note that since $b_m^k(i) \neq X_i$, $i \in \mathrm{supp}(C^k)$. Let $r < s-1$. There is a $j < k$ such that $p_r = p_j$; so we need to show that $b_m^{p_j}(i) \supseteq_w b_m^k(i)$. Let $\alpha = \mathrm{hccp}(b_m^{p_j}(i), b_m^k(i))$. By Claim 2 and by definition of cinch-sets, $\alpha \leqslant \mathrm{hccp}(b_m^{p_j}, b_m^k) \leqslant \gamma_j$. Then

$$\begin{array}{lll}
b_m^k & \subseteq \mathrm{pr}_{C^k}\big(\pi_{\gamma_j}(b_m^{p_j})\big) & \text{(by definition of } b_m^k) \\
\Rightarrow b_m^k(i) & \subseteq \mathrm{pr}_{C^k}\big(\pi_{\gamma_j}(b_m^{p_j})\big)(i) & \\
\Rightarrow b_m^k(i) & \subseteq \pi_{\gamma_j}(b_m^{p_j})(i) & \text{(as } i \in \mathrm{supp}(C^k)) \\
\Rightarrow \pi_\alpha(b_m^k(i)) & \subseteq \pi_\alpha\big(\pi_{\gamma_j}\big(b_m^{p_j}(i)\big)\big) & \text{(by Lemma 2.2)} \\
\Rightarrow \pi_\alpha(b_m^k(i)) & \subseteq \pi_\alpha\big(b_m^{p_j}(i)\big) & \text{(by Lemma 2.2)} \\
\Rightarrow b_m^k(i) & \subseteq_w b_m^{p_j}(i), &
\end{array}$$

as desired. This proves Claim 5. □

Claim 6. *For all $m < \omega$, $\bigcap_{p\leqslant k} b_m^p \neq \emptyset$.*

Proof. Let $m < \omega$ and $i \in I$; we must prove that $\bigcap_{p \leqslant k} b_m^p(i) \neq \emptyset$. Note that

$$\bigcap_{p \leqslant k} b_m^p(i) = \bigcap \{ b_m^{p_j}(i) \colon b_m^{p_j}(i) \neq X_i \}.$$

Let $\{p_r \colon r < s\}$ list, in increasing order, the set $\{p_j \colon b_m^{p_j}(i) \neq X_i\}$. By Claim 5, $\{b_m^{p_r}(i) \colon r < s\}$ is weakly-descending. Since (by construction) each $b_m^{p_r}(i)$ is nonempty, by Lemma 2.10 we have

$$\emptyset \neq \bigcap \{ b_m^{p_r}(i) \colon r < s \} = \bigcap \{ b_m^{p_j}(i) \colon b_m^{p_j}(i) \neq X_i \} = \bigcap_{p \leqslant k} b_m^p(i).$$

This proves Claim 6. □

Thus Lemma 3.2 is proved. □

Theorem 3.3. *Let $n < \omega$ and let $C^0, C^1, \ldots, C^{n-1}$ be maximal antichains. Then for all but finitely-many $m < \omega$, $[\bigcap_{k<n} F^{C^k}] \cap U_m \neq \emptyset$.*

Proof. Let $2 \leqslant n < \omega$. Let $C^0, C^1, \ldots, C^{n-1}$ be maximal antichains, and without loss of generality assume $\xi^{C^0} \leqslant \xi^{C^1} \leqslant \cdots \leqslant \xi^{C^{n-1}}$. For each $m < \omega$, set $b_m^0 = c_0(m, 0)$. Then conditions (1)–(4) in the hypotheses of Lemma 3.2 are clearly satisfied for the sequence $\{b_m^0 \colon m < \omega\}$. By repeatedly applying Lemma 3.2 we may obtain, for each $k < n$, sequences $\{b_m^k \colon m < \omega\}$ and natural numbers N_k such that

(1) for all $k < n$, there exists $S_k \geqslant \max\{S_p \colon p < k\}$ such that for all $m \geqslant S_k$, there exists $l < f_{\xi^{C^k}_{N_k}}(m)$ such that $\emptyset \neq b_m^k \subseteq c_k(m, l)$; and
(2) For all $m < \omega$, $\bigcap_{k<n} b_m^k \neq \emptyset$.

Pick $N \geqslant S_{n-1}$ so large that for all $m > N$ and $k < n$, $f_{\xi^{C^k}_{N_k}}(m) \leqslant f_{\xi^{C^k}}(m)$. Let $m > N$ and $k < n$. Then by (1) and by choice of N, there is an $l < f_{\xi^{C^k}_{N_k}}(m) \leqslant f_{\xi^{C^k}}(m)$ such that $\emptyset \neq b_m^k \subseteq c_k(m, l)$; so by (2),

$$(\forall m > N)(\forall k < n)\left(\emptyset \neq \bigcap_{p<n} b_m^p \subseteq \bigcup_{l < f_{\xi C_k}(m)} c_k(m, l) \right).$$

Recall that we defined the filter element F^{C^k} corresponding to the maximal antichain C^k by

$$F^{C^k} = \bigcup_{m<\omega} \left(\bigcup_{l < f_{\xi C_k}(m)} c_k(m, l) \right).$$

Thus for each $m > N$,

$$\emptyset \neq \bigcap_{k<n} b_m^k \subseteq \left[\bigcap_{k<n} F^{C^k} \right] \cap U_m. \qquad \square$$

Thus $\mathcal{F}$ generates a nice remote filter with respect to $\{U_m \colon m < \omega\}$, and therefore:

Theorem 3.4. (CH) *If I is any index set and $X = \prod_{i \in I} X_i$ is a product of completely regular, ccc spaces each of weight ω_2; and if X is ccc and non-pseudocompact*; *then X has remote points.*

4. Questions

(1) Under CH, does every ccc non-pseudocompact space have remote points?
(2) In particular, under CH, does every ccc non-pseudocompact space of weight at most ω_3 have remote points?

(3) In the statement of Theorem 3.4, is it necessary to assume that X is ccc?
(4) Suppose that it were known, under CH, that every ccc non-pseudocompact space of weight κ had remote points (for some $\kappa > \mathfrak{c}$). Would it then follow that any product of such spaces that was also ccc and non-pseudocompact had remote points?

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